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NUMERICAL SIMULATION OF WATER WAVE BASED ON CHEBYSHEV SPECTRAL METHOD

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ABSTRACT

A water wave model was presented by tidal wave partial differential equations in the form of two-dimension, nondimensionalization, and matrix. Numerical simulation of water wave adopted Chebyshev spectral method to solve the partial differential equations. As an example，propagation of a water wave in a square pool due to a water column perturbation was simulated. The simulation result agrees with wave diffusion rule and numerically verified by the same amount of water in the pool. The result showed that the Chebyshev spectral method is concise and accurate in numerical simulation of water wave.

KEYWORDS: *Numerical Simulation, Chebyshev Spectral Method Water Wave Equations*

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INTRODUCTION

Numerical simulation provides an effective means to study water wave morphology. Having a detailed understanding of water wave is significant when considering some ocean problems[1-3]. The tidal wave equations characterize the motion of water waves in the flow field by expressing the velocity of water flow and the height of the water level. The Lagrangian method [4, 5] and the Euler method [6] are the two methods constructing the tidal wave equations based on fluid mechanics. The idea of the Lagrangian method is to descript the movement of particles. That is, based on the study of the motion process of a single fluid particle, the motion of all the particles is integrated to constitute the movement of the entire fluid. The Euler method studies fixed spatial points. By observing the change of the moving elements over time at each fixed space point in the flow space, the movement of the entire fluid is obtained by combining enough spatial points. The tidal wave equations established by the Euler method is studied in this paper.

In the Cartesian coordinate system, the two-dimensional water wave equations are obtained by the vertical integral of the three-dimensional water wave equations[7, 8]. As partial differential terms of time and space is contained in water wave equations, it is difficult to obtain an analytical solution. Therefore, numerical solutions are needed by choosing a numerical calculation method. Spectral method is one of three tools for solving partial differential equations, along with the difference method and the finite element method[9-11]. The application of spectral methods is mainly in the fields of solving non-linear heat conduction equation and fluid mechanics equation, predicting numerical weather prediction and so on[12]. In recent years, the spectral method has been developed mainly in two directions. The mathematical research focuses on the methodology and the solution of various partial differential equations and its convergence ([13]), while the physical research is mainly to solve various physical problems by using a spectral method based on Fourier or Chebyshev polynomial as the basis function[14, 15]. The advantages of the spectral method are its infinite order convergence and conveniently applied the fast Fourier transfer[16-18].

In this paper, the Chebyshev spectral method is used to solve two-dimensional water wave equations and obtain a numerical simulation of a water wave in a square pool. The result can provide a reference for the numerical solution of water wave equations.

Water Wave Model

The water wave equations give the velocity of water wave motion and the height of the water level. Two-dimensional water wave equations and its boundary conditions in the Cartesian coordinated system is as following[8],

$$\begin{cases} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}[(\xi+h)u] + \frac{\partial}{\partial y}[(\xi+h)v] = 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial \xi}{\partial x} - v + g\frac{u\sqrt{u^2+v^2}}{(\xi+h)c_f^2} = 0, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial \xi}{\partial x} + u + g\frac{v\sqrt{u^2+v^2}}{(\xi+h)c_f^2} = 0 \end{cases} \quad \begin{cases} u\big|_{nf} = 0 \\ \frac{\partial \xi}{\partial n}\Big|_{nf} = 0 \end{cases} = \tag{1}$$

where u and v represent respectively the average flow velocity of the water current in the x and y-direction, t the time variable, h the average water depth, ξ the water level, g the gravitational acceleration, c_f the roughness coefficient or chezy coefficient ($c_f = (h+\xi)^{1/6} / n$, n the Manning coefficient which value is taken as 0.003).

For the convenience of calculation, all the variables of Eq. 1 have been nondimensionalized[8]. And all the variables can be restored to standard dimensions by

$$\begin{aligned} &\bar{g} = gXf^2, \quad \bar{c}_f = c_f X^{1/2} f, \quad \bar{t} = tf^{-1}, \quad \bar{x} = xX, \\ &\bar{y} = yX, \quad \bar{\xi} = \xi x, \quad \bar{h} = hX, \quad \bar{u} = uXf^{-1}, \quad \bar{v} = vXf^{-1} \end{aligned} \tag{2}$$

where X is the normalized length. f the Coriolis coefficient ($f = 2\omega \sin\phi$, ω is the Earth's rotation rate, ϕ is latitude).

Chebyshev Spectral Method

The Chebyshev spectral method is used to solve the spatial partial differential term in Eq. 1. The domain of the Chebyshev polynomial is a square region of $[-1,1]\times[-1,1]$, which is divided by $N\times N$ small regions by non-equal intervals. The mesh nodes in the regions are taken as the Chebyshev points, stated as follows[19]:

$$\begin{cases} x_j = \cos(\frac{j\pi}{N}), \ j = 0,\ 1,\ \ldots,\ N \\ y_i = \cos(\frac{i\pi}{N}), \ i = 0,\ 1,\ \ldots,\ N \end{cases} \tag{3}$$

where i and j is the index of the Chebyshev points along x and y-axis, respectively. Then Chebyshev points are used to construct Chebyshev differentiation matrix. For each $N\geq 1$, The Chebyshev differential matrix D_N can be shown below

Table 1

	$\frac{2N^2+1}{6}$		$2\frac{(-1)^j}{1-x_j}$		$\frac{1}{2}(-1)^N$
				$\frac{(-1)^{i+j}}{x_i-x_j}$	
$D_N =$	$-\frac{1}{2}\frac{(-1)^i}{1-x_i}$		$\frac{-x_j}{2(1-x_j^2)}$		$\frac{1}{2}\frac{(-1)^{N+i}}{1+x_i}$
		$\frac{(-1)^{i+j}}{x_i-x_j}$			
	$-\frac{1}{2}(-1)^N$		$-2\frac{(-1)^{N+j}}{1+x_j}$		$-\frac{2N^2+1}{6}$

Thus, the first-order partial differential matrix D_x and D_y can be expressed as

$$\begin{cases} D_x = D_N \otimes I \\ D_y = I \otimes D_N \end{cases} \tag{5}$$

Where D_N is the Chebyshev differential matrix, ⊗ the Kronecker product, I the identity matrix. Defining that V is an N+1 order matrix composed of function values on the mesh nodes, the first-order partial derivatives of the spatial variables over x and y can be expressed as:

$$\begin{cases} V_x' = D_x * V \\ V_y' = D_y * V \end{cases} \tag{6}$$

Where V in the calculation is taken as a column shape $V(:)$. It means in all the matrix multiplication related to D_N, D_x, D_y, Matrix variable V should be converted to column shape in the following way:

$$V = \begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & & \vdots \\ V_{N1} & \cdots & V_{NN} \end{bmatrix} \quad \Rightarrow$$
$$V(:) = [V_{11} \ \ldots \ V_{N1} \ \ldots \ V_{1N} \ \ldots \ V_{NN}]^T \tag{7}$$

To numerically solve Eq. 1, the partial differential term over variables x and y is given by the Chebyshev differential matrix according to Eq. 6, while the partial differential term of the time variable t is given by a leapfrog formula. Thus, the numerical solution of Eq. 1 is the follows:

$$\begin{cases} \xi_{n+1} = \xi_n - \Delta t D_x[(\xi_n + h)\cdot u_a] - \Delta t D_y[(\xi_n + h)\cdot v_a] \\ u_{n+1} = u_a + \Delta t v_a - \Delta t g(D_x \xi_{n+1}) \\ v_{n+1} = v_a - \Delta t u_a - \Delta t g(D_y \xi_{n+1}) \end{cases} \tag{8}$$

where ξ_{n+1} is the water level at n+1 time sequence, u_{n+1} and v_{n+1} the wave velocity in direction of x and y-axis, respectively. These three variables are matrixes of N+1 order and get from u_n, v_n and ξ_n at n time sequence. The u_a and v_a is auxiliary speed stated by

$$\begin{cases} u_a = u_n - \Delta t[u_n \cdot (D_x u_n) + v_n \cdot (D_y u_n) + C_f \cdot \dfrac{u_n \sqrt{(u_n^2 + v_n^2)}}{(\xi_n + h)}] \\ v_a = v_n - \Delta t[u_n \cdot (D_x v_n) + v_n \cdot (D_y v_n) + C_f \cdot \dfrac{v_n \sqrt{u_n^2 + v_n^2)}}{(\xi_n + h)}] \end{cases} \quad (9)$$

Numerical Experiment

By using the Chebyshev spectral method to solve two-dimensional water wave equations, water wave caused by a water column perturbation in a uniform depth square pool was simulated. In the simulation, assuming the square wall and the bottom surface of the square pool are smooth and the ratio of depth to length of the pool is less than 1/1000, where the water depth is taken as 3.8 m and the side length is 4000 *m*. Chebyshev mesh node takes 32×32 At the initial moment t = 0, the water column perturbation is close to the corner of the pool and has a height of 0.6 m. The time step for numerical calculation Δt is 0.5 s. The calculation accuracy is 10^{-5} m, that is when the mean value of the water level difference calculated twice before and after each grid point is less than 10^{-5} m, it is considered that the water wave in the pool tends to be stable and the calculation is stopped.

The duration of the simulation process continues 100.92 hours, the fluctuation of water in the pool tends to be stable. The water level at each grid point is almost the same, and the velocity vector is approximately zero.

Fig. 1(a) and Fig. 2(a) show the initial state for the simulation, where the water flow velocity is zero and the perturbation water column is 0.6 m high near the corner. Fig. 1(b) and Fig. 2(b) shows the perturbation water column falls and the water spreads in the pool at t = 85 *s*. From the diagram of the velocity vector in Fig. 1(b), it can be seen that the water velocity vector is evenly arranged in a circle. Fig. 1(c) and Fig. 2(c) shows the superimposition of the forward wave and the reflected echo wave in the pool when the perturbation water column falls for 545 s, and the alternation of water waves with peaks and troughs. From Figs. 1-2, it can be seen that the water column perturbation generates water waves and water wave spreads around under the action of gravity. In the propagation, water waves hit the wall and echo waves formed. The echo wave is superimposed on the forward wave, and the both cancel each other out. After water wave collides with the wall several times, the water surface of the pool calmed down. The simulation results basically conform to the physical law of water wave motion.

In order to obtain quantitative verification of the simulated results, the initial value of the average water level rise caused by the water column perturbation and the terminal value is calculated shown in Table 1. The theoretical value of the rise in water level is the water level change obtained by calculating the water level rise caused by the volume of the perturbed water column, while the experimental value is the numerical result of adapting the Chebyshev spectral method to solve the flow field of the square pool with water column perturbation. The error is smaller than the calculation accuracy taken as 5×10^{-4} *m*, indicating the Chebyshev spectral method is accurate and useful in the water wave simulation.

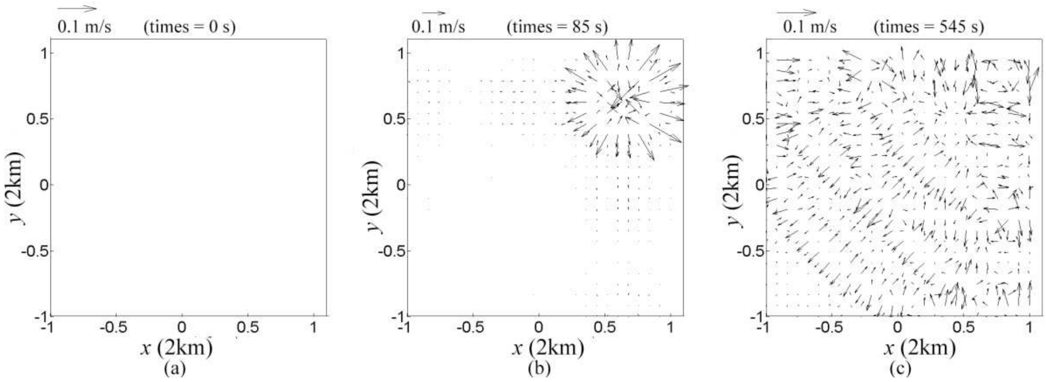

Figure 1: Velocity of the Water Wave in the Pool

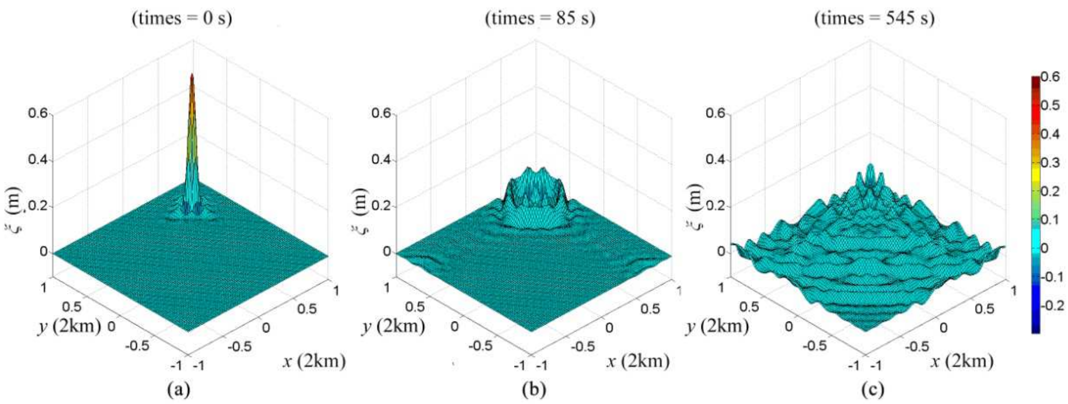

Figure 2: Level of the Water Wave in the Pool

Table 1: Average Water Level Rise and its Absolute Error

Water Level Rise (Theoretical Value) [M]	Water Level Rise (Experimental Value) [M]	Absolute Error
8.6×10^{-4}	8.3×10^{-4}	0.3×10^{-4}

CONCLUSIONS

In this paper, A matrix solution form of two-dimensional tidal wave equations based on the Chebyshev spectral method is presented. In the numerical solution process, the partial differential term on the spatial variable is represented by the Chebyshev differential matrix, and the partial differential time variable is treated by a leap frag formula. Taking a square water column perturbation at the initial stage as a numerical experiment, the results about the simulation of water wave are achieved, and the simulation results are consistent with the water wave diffusion law. The Chebyshev spectral method provides a clear and simple matrix format for numerical solving the two-dimensional water wave equations.

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